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Exponential Stability of a Class of Neural Networks with Discrete Time-varying and Distributed Delays*

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Abstract: The paper is devoted to the exponential stability of a class of neural networks with both discrete time-varying and distributed delays. In virtue of nonlinear measure, a delay-independent sufficient condition is derived for the existence, uniqueness and exponential stability of the equilibrium point. Since assumptions on boundedness, monotonicity and differentiability of activation functions and differentiability of time-varying transmission delay functions are avoided, the new stability criterion is an extension of some existing results. Moreover, an additional merit of the method is to provide the exponentially convergent velocity of the solutions. Finally, an example is provided to illustrate effectiveness of the method.

Keywords: exponentially stability; neural networks; distributed delays; time-varying delays; nonlinear measure

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1 Introduction

Neural networks have been widely investigated due to their potential applications such as associative memory, parallel computing, classification, pattern recognition and combinatorial optimization. Since the stability analysis of the neural networks is prerequisite in such applications and time delays are unavoidably encountered in their implementation, it is necessary to study stability of neural networks with different types of delays. Up to now, there exist many excellent results on stability of neural networks with discrete time-varying and distributed delays by a linear matrix inequality approach[1-6], constructing Lyapunov functions[7-10] and the M-matrix theory[11], respectively. However, the boundedness of activation functions is usually imposed in the linear matrix inequality approach to guarantee the existence of equilibrium point of the neural networks. As we all know, the construction of a proper Lyapunov function becomes usually very skill, that is, there exists no general rule to guide how to construct a suitable Lyapunov function for a given system. Compared with the above two approaches, M-matrix approach requires neither the boundedness of activation functions nor the construction

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of Lyapunov functions, but it doesn't provide convergent velocity of the solutions. In some applications, it is usually necessary to estimate the convergent velocity of the solutions.

Motivated by this, the aim of the paper is to investigate exponential stability of a class of neural networks with both discrete time-varying and distributed delays by nonlinear measure method[12,13]. Since the nonlinear measure is a nonlinear extension of matrix measure, our derived criterion is an essential characterization to stability of nonlinear systems with delays and it is worth mentioning that our method provides the exponentially convergent velocity of the solutions. Moreover, compared with some existing methods, our method does not require the differentiability of time-varying transmission delay functions, the symmetry of connection matrices and the differentiability, monotonicity and boundedness of activation functions, which means that our stability criterion is less conservative.

2 Model description and assumptions

We consider the following model of neural networks with discrete time-varying and distributed delays

$$\begin{aligned}\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i\big(u_i(t)\big)\Big[b_i\big(u_i(t)\big) - \sum_{j=1}^{n} c_{ij} f_j\big(u_j(t)\big) - \sum_{j=1}^{n} d_{ij} g_j\big(u_j(t-\tau_{ij}(t))\big) \\ - \sum_{j=1}^{n} q_{ij} \int_0^{\infty} K_{ij}(t-s) h_j\big(u_j(s)\big)\mathrm{d}s + I_i\Big] \end{aligned} \tag{1}$$

for $i = 1, 2, \cdots, n$. Here $n \geq 2$ is the number of neurons in the networks; u_i is the state variable of the ith neuron, a_i represents the amplification function, b_i denotes the appropriately behaved function such that the solutions of (1) are bounded; f_j, g_j and h_j are the activation functions; $\tau_{ij}(t)$ corresponds to the transmission delay along the axon of the jth unit from the ith unit and satisfies $b = \sup\{\tau_{ij}(t) : 1 \leq i, j \leq n, t \in \mathbb{R}\} < \infty$; $C = (c_{ij})_{n\times n}$, $D = (d_{ij})_{n\times n}$ and $Q = (q_{ij})_{n\times n}$ are connection matrices; I_i is the constant external input and K_{ij} is the delay kernel. The initial condition associated with the model (1) satisfies

$$u_i = \phi_i \in C\big((-\infty, 0], \mathbb{R}\big), \quad i = 1, 2, \cdots, n,$$

where $C((-\infty,0],\mathbb{R})$ denotes the set of all continuous functions from $(-\infty,0]$ to $\mathbb{R}$.

Throughout this paper, we make the following assumptions:

(H_1): Each a_i is continuous and satisfies $0 < \grave{\alpha} \leq a_i(s) \leq \acute{a}$, for all $s \in \mathbb{R}$;

(H_2): Each b_i is continuous and there exists a constant $\lambda_i > 0$ such that

$$(x-y)\big[b_i(x) - b_i(y)\big] \geq \lambda_i (x-y)^2, \quad \forall\, x, y \in \mathbb{R}, \quad i = 1, 2, \cdots, n;$$

(H_3): f_j, g_j and h_j are Lipschitz continuous for $j = 1, 2, \cdots, n$. Conveniently, the following constants

$$L(f_j) = \sup_{x,y\in\mathbb{R}, x\neq y} \frac{\|f_j(x) - f_j(y)\|}{\|x-y\|}, \quad L(g_j) = \sup_{x,y\in\mathbb{R}, x\neq y} \frac{\|g_j(x) - g_j(y)\|}{\|x-y\|},$$

$$L(h_j) = \sup_{x,y\in\mathbb{R}, x\neq y} \frac{\|h_j(x) - h_j(y)\|}{\|x-y\|},$$

are called minimal Lipschitz constants(MLC) of f_j, g_j and h_j respectively;

(H_4): $K_{ij} : [0, +\infty) \to [0, +\infty)$ is a continuous function and satisfies

$$\int_0^{\infty} \mathrm{e}^{\beta s} K_{ij}(s)\mathrm{d}s = p_{ij}(\beta),$$

where p_{ij} is a continuous function on $[0, \delta)$ for some $\delta > 0$, and $p_{ij}(0) = 1,\ i, j = 1, 2, \cdots, n$.

3 Preliminaries

Let n-dimensional real vector space $\mathbb{R}^n$ be endowed with 1-norm $\|\cdot\|_1$ defined by

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \forall\, x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n,$$

where the superscript "T" denotes the transpose. Let $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathbb{R}^n$ and $\mathrm{sign}(x) = (\mathrm{sign}(x_1), \mathrm{sign}(x_2), \cdots, \mathrm{sign}(x_n))^T$, the sign vector of $x \in \mathbb{R}^n$, where $\mathrm{sign}(r)$ represents the sign function of $r \in \mathbb{R}$. It is obvious that $\|x\|_1 = \langle x, \mathrm{sign}(x) \rangle$ and $\|x\|_1 \geq \langle x, \mathrm{sign}(y) \rangle$ hold for all x, $y \in \mathbb{R}^n$.

Consider the following time-delayed system

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{dt} = F(u(t)) + G(u_t(s)), \quad u(t) \in \Omega, \quad t \geq 0, \quad s \in (-\infty, 0], \\ u_0 = \phi \in C((-\infty, 0], \Omega), \end{cases} \tag{2}$$

where F and G are nonlinear operators from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $u_t \in C((-\infty, 0], \Omega)$ is defined by

$$u_t(s) = u(t+s), \quad \|u_t\|_\infty = \sup_{-\infty < s \leq 0} \|u(t+s)\|_1,$$

$$F\big(u(t)\big) = \big(F_1(u(t)), F_2(u(t)), \cdots, F_n(u(t))\big)^T,$$

$$G\big(u_t(s)\big) = \big(G_1(u_t(s)), G_2(u_t(s)), \cdots, G_n(u_t(s))\big)^T, \quad \forall\, t \geq 0, \quad s \in (-\infty, 0].$$

Definition 1[12] Assume that Ω is an open subset of $\mathbb{R}^n$ and F is a nonlinear operator from Ω into $\mathbb{R}^n$. The constant

$$m_\Omega(F) = \sup_{x, y \in \Omega, x \neq y} \frac{\langle F(x) - F(y), \mathrm{sign}(x-y) \rangle}{\|x - y\|_1}$$

is called the nonlinear measure of F on Ω.

Definition 2 Let u^* be an equilibrium point of the system (2) and Ω an open neighborhood of u^*. Then u^* is said to be exponentially stable on Ω if there exist two positive constants α and M such that the solution u of (2) initiated form any $\phi \in C((-\infty, 0], \Omega)$ satisfies

$$\|u(t) - u^*\|_1 \leq M\mathrm{e}^{-\alpha t} \sup_{-\infty < s \leq 0} \|\phi(s) - x^*\|_1, \quad t \geq 0.$$

Moreover, if u^* is exponentially stable on the whole space $\mathbb{R}^n$, the system (2) is said to be globally exponentially stable.

Lemma 1[12] If $m_\Omega(F) < 0$, then $F : \Omega \to \mathbb{R}^n$ is one-to-one mapping. In addition, if $\Omega = \mathbb{R}^n$, then F is a homeomorphism of $\mathbb{R}^n$.

Paper [12] investigated the following discrete-delay differential system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F\big(u(t)\big) + G\big(u_\tau(t)\big), \tag{3}$$

where F and G are mapping from Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$ and $u_\tau(t)$ is defined as

$$G_i\big(u_\tau(t)\big) = G_i\Big(\big(u_1(t-\tau_{i1}(t)), u_2(t-\tau_{i2}(t)), \cdots, u_n(t-\tau_{in}(t))\big)^T\Big),$$

where $G(u) = (G_1(u), G_2(u), \cdots, G_n(u))^T$.

Lemma 2[12] Let Ω be a neighborhood of the equilibrium x^* of the system (3). If the inequality

$$m_{A^{-1}(\Omega)}(FA) + L_{A^{-1}(\Omega)}(GA) < 0, \tag{4}$$

holds for some diagonal matrix $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)^T$ with $a_i > 0$, then x^* is exponentially stable on Ω, where $L_{A^{-1}(\Omega)}(GA)$ denotes MLC of nonlinear Lipschitz operator GA on its domain $A^{-1}(\Omega)$. Moreover, the exponential decay estimation of any solution $x(t)$ of the system (2) initiated from $x_0 \in C([-b, 0], \Omega)$ is determined by

$$\|x(t) - x^*\|_1 \le \mathrm{e}^{-\lambda t} \cdot \sup_{-b \le s \le 0} \|x_0(s) - x^*\|_1, \quad \forall\, t \ge 0, \tag{5}$$

where λ is the unique positive solution of the equation

$$0 = \lambda \cdot \min_{1 \le i \le n} a_i + m_{A^{-1}(\Omega)}(FA) + L_{A^{-1}(\Omega)}(GA) \cdot \mathrm{e}^{b\lambda}. \tag{6}$$

Remark 1 By revising the proof of Lemma 2 in [12] we can easily conclude that if the inequality (4) holds under the assumptions (H_1)-(H_4), the solution of the system (2) initiated from $x_0 \in C((-\infty, 0], \Omega)$ is exponentially stable and has the following exponential decay estimation

$$\|x(t) - x^*\|_1 \le \mathrm{e}^{-\lambda t} \cdot \sup_{-\infty < s \le 0} \|x_0(s) - x^*\|_1, \quad \forall\, t \ge 0, \tag{7}$$

where λ is the unique positive solution of the equation (6). It should be mentioned that the equality

$$\int_{-\infty}^{t} K_{ij}(t-s)\mathrm{d}s = 1, \quad \forall\, t \ge 0, \tag{8}$$

was used in the revising proof. The equality (8) is easily derived from (H_4).

4 Main results

In this section, we firstly prove that the system (1) enjoys a unique equilibrium point in Ω. Let Ω_i denote the projection of the open subset Ω on the i-th axis of $\mathbb{R}^n$.

Theorem 1 Suppose that (H_1)-(H_4) hold and there exist positive real numbers d_i ($i = 1, 2, \cdots, n$) such that

$$\max_{1 \le j \le n} \frac{1}{\lambda_j}\Big\{L(f_j)\sum_{i=1}^{n}\frac{d_j}{d_i}|c_{ij}| + L(g_j)\sum_{i=1}^{n}\frac{d_j}{d_i}|d_{ij}| + L(h_j)\sum_{i=1}^{n}\frac{d_j}{d_i}|q_{ij}|\Big\} < 1, \tag{9}$$

hold, where $L(f_j)$, $L(g_j)$ and $L(h_j)$ denote MLC of f_j, g_j and h_j on Ω_j, respectively. Then for each set of external inputs I_i, the model (1) has a unique equilibrium point u^* in Ω.

Proof Define $P = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ and an operator $H : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H_i(u) = -\Big[b_i(u_i) - \sum_{j=1}^{n} c_{ij} f_j(u_j) - \sum_{j=1}^{n} d_{ij} g_j(u_j) - \sum_{j=1}^{n} q_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j(u_j) \mathrm{d}s + I_i\Big],$$

where

$$u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n, \quad H(u) = \big(H_1(u), H_2(u), \cdots, H_n(u)\big)^T.$$

From the assumption (H_1) we easily derive that $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ is an equilibrium point of the model (1) if and only if $H(u^*) = 0$.

For x, $y \in P^{-1}(\Omega)$, we have

$$\begin{aligned}
&\big\langle P^{-1}H(Px) - P^{-1}H(Py), \mathrm{sign}(x-y)\big\rangle \\
&= \sum_{i=1}^{n} \mathrm{sign}(x_i - y_i)\Big\{ - d_i^{-1}\big(b_i(d_i x_i) - b_i(d_i y_i)\big) + \sum_{j=1}^{n} d_i^{-1} c_{ij}\big(f_j(d_j x_j) - f_j(d_j y_j)\big) \\
&\quad + \sum_{j=1}^{n} d_i^{-1} d_{ij}\big(g_j(d_j x_j) - g_j(d_j y_j)\big) + \sum_{j=1}^{n} d_i^{-1} q_{ij} \int_{-\infty}^{t} K_{ij}(t-s)\big(h_j(d_j x_j) - h_j(d_j y_j)\big)\mathrm{d}s\Big\} \\
&\le \sum_{i=1}^{n} d_i^{-1}\Big\{ - |b_i(d_i x_i) - b_i(d_i y_i)| + \sum_{j=1}^{n} \big(|c_{ij}||f_j(d_j x_j) - f_j(d_j y_j)| \\
&\quad + |d_{ij}||g_j(d_j x_j) - g_j(d_j y_j)| + |q_{ij}| \int_{-\infty}^{t} K_{ij}(t-s)|h_j(d_j x_j) - h_j(d_j y_j)|\mathrm{d}s\big)\Big\} \\
&\le -\sum_{i=1}^{n} \lambda_i |x_i - y_i| + \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{d_j}{d_i}\Big[|c_{ij}|L(f_j)|x_j - y_j| + |d_{ij}|L(g_j)|x_j - y_j| \\
&\quad + |q_{ij}|L(h_j)|x_j - y_j| \int_{-\infty}^{t} K_{ij}(t-s)\mathrm{d}s\Big] \\
&= -\sum_{i=1}^{n} \lambda_i |x_i - y_i| + \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{d_j}{d_i}\big[\,|c_{ij}|L(f_j) + |d_{ij}|L(g_j) + |q_{ij}|L(h_j)\big]|x_j - y_j| \\
&= -\sum_{j=1}^{n}\Big\{\lambda_j - \sum_{i=1}^{n} \frac{d_j}{d_i}\big[\,|c_{ij}|L(f_j) + |d_{ij}|L(g_j) + |q_{ij}|L(h_j)\big]\Big\}.
\end{aligned} \tag{10}$$

The combination of (10) and (9) implies that $m_{P^{-1}(\Omega)}(P^{-1}HP) < 0$. In the light of Lemma 1, $P^{-1}HP$ is one-to-one. Consequently, there is only $v \in \mathbb{R}^n$ such that $P^{-1}HP(v) = 0$. The equilibrium point u^* of the model (1) is unique in Ω since P is non-singular.

In what follows, we give an exponential stability criterion of the system (1) by Theorem 1 and Lemma 2. For this, let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$F_i(u) = -b_i(u_i) + \sum_{j=1}^{n} c_{ij} f_j(u_j)$$

and $G:\mathbb{R}^n\to\mathbb{R}^n$ by

$$G_i(u)=\sum_{j=1}^{n}d_{ij}g_j(u_j)+\sum_{j=1}^{n}q_{ij}\int_{-\infty}^{t}K_{ij}(t-s)h_j(u_j)\mathrm{d}s+I_i.$$

Theorem 2 Suppose that (H_1)-(H_4) hold. Assume u^* to be an equilibrium point of the model (1) and Ω a neighborhood of u^*. If there exist a set of positive real numbers d_i $(i=1,2,\cdots,n)$ such that the condition (9) holds, then for each set of external input I_i, the model (1) is exponentially stabile on Ω. Particularly, if $u(t)$ is the solution of the model (1) initiated from $\phi\in C((-\infty,0],\Omega)$, then

$$\|u(t)-u^*\|_1\le \mathrm{e}^{-\sigma t}\cdot\frac{\max_{1\le i\le n}d_i}{\min_{1\le i\le n}d_i}\cdot\sup_{-\infty<s\le 0}\|\phi(s)-u^*\|_1,\quad \forall\, t\ge 0,\tag{11}$$

where σ is the unique positive solution of the equation

$$\sigma\cdot\min_{1\le j\le n}c_j^{-1}-1+k\,\mathrm{e}^{b\sigma}=0,\tag{12}$$

with

$$c_j=\lambda_j-L(f_j)\sum_{i=1}^{n}\frac{d_j}{d_i}|c_{ij}|$$

and

$$k=\max_{1\le j\le n}\Big\{c_j^{-1}\sum_{i=1}^{n}\frac{d_j}{d_i}\big[\,|d_{ij}|L(g_j)+|q_{ij}|L(h_j)\big]\Big\}.$$

Proof Let $P=\mathrm{diag}(d_1,d_2,\cdots,d_n)$ and $A=\mathrm{diag}(c_1^{-1},c_2^{-1},\cdots,c_n^{-1})$. It immediately follows from (9) that

$$c_j=\lambda_j-L(f_j)\sum_{i=1}^{n}\frac{d_j}{d_i}|c_{ij}|>0,\quad j=1,2,\cdots,n.$$

For all $x,\,y\in A^{-1}P^{-1}(\Omega)$,

$$\begin{aligned}&\big\langle P^{-1}F(PAx)-P^{-1}F(PAy),\mathrm{sign}(x-y)\big\rangle\\
\le&\sum_{i=1}^{n}d_i^{-1}\Big\{-|b_i(d_ic_i^{-1}x_i)-b_i(d_ic_i^{-1}y_i)|+\sum_{j=1}^{n}|c_{ij}||f_j(d_jc_j^{-1}x_j)-f_j(d_jc_j^{-1}y_j)|\Big\}\\
\le&\sum_{i=1}^{n}d_i^{-1}\Big\{-d_ic_i^{-1}\lambda_i|x_i-y_i|+\sum_{j=1}^{n}|c_{ij}|L(f_j)d_jc_j^{-1}|x_j-y_j|\Big\}\\
=&-\sum_{j=1}^{n}c_j^{-1}\Big(\lambda_j-L(f_j)\sum_{i=1}^{n}\frac{d_j}{d_i}|c_{ij}|\Big)|x_j-y_j|=-\|x-y\|_1,\end{aligned}$$

which implies that $m_{A^{-1}P^{-1}(\Omega)}(P^{-1}FPA) \le -1$. For all $x,\, y \in A^{-1}P^{-1}(\Omega)$, we have

$$
\begin{aligned}
\left\|P^{-1}G(PA)x - P^{-1}G(PA)y\right\|_1 &= \sum_{i=1}^{n}\Big| d_i^{-1}\Big\{\sum_{j=1}^{n} d_{ij}\left[g_j(d_jc_j^{-1}x_j) - g_j(d_jc_j^{-1}y_j)\right] \\
&\quad + \sum_{j=1}^{n} q_{ij}\int_{-\infty}^{t} K_{ij}(t-s)\left[h_j(d_jc_j^{-1}x_j) - h_j(d_jc_j^{-1}y_j)\right]\mathrm{d}s\Big\}\Big| \\
&\le \sum_{j=1}^{n}\Big\{c_j^{-1}\sum_{j=1}^{n}\frac{d_j}{d_i}\left[\,|d_{ij}|L(g_j) + |q_{ij}|L(h_j)\right]\Big\}|x_j - y_j|,
\end{aligned}
$$

thus

$$
L_{A^{-1}P^{-1}(\Omega)}(P^{-1}GPA) \le \max_{1\le j\le n}\Big\{c_j^{-1}\sum_{i=1}^{n}\frac{d_j}{d_i}\left[\,|d_{ij}|L(g_j) + |q_{ij}|L(h_j)\right]\Big\} = k.
$$

Consequently, from (9) we conclude that

$$
\begin{aligned}
& m_{A^{-1}P^{-1}(\Omega)}(P^{-1}FPA) + L_{A^{-1}P^{-1}(\Omega)}(P^{-1}GPA) \\
\le\ & -1 + \max_{1\le j\le n}\Big\{c_j^{-1}\sum_{i=1}^{n}\frac{d_j}{d_i}\left[\,|d_{ij}|L(g_j) + |q_{ij}|L(h_j)\right]\Big\} \\
=\ & \max_{1\le j\le n}\left\{\frac{-\lambda_j + \sum_{i=1}^{n}\frac{d_j}{d_i}|c_{ij}|L(f_j) + \sum_{i=1}^{n}\frac{d_j}{d_i}\left[\,|d_{ij}|L(g_j) + |q_{ij}|L(h_j)\right]}{\lambda_j - \sum_{i=1}^{n}\frac{d_j}{d_i}|c_{ij}|L(f_j)}\right\} < 0.
\end{aligned}
$$

From Remark 1 it is derived that the solution $x(t)$ with the initial function $x \in C((-\infty,0],\Omega)$ of the time-delayed system

$$
\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F\big(x(t)\big) + P^{-1}GP\big(x_t(s)\big), \tag{13}
$$

satisfies

$$
\|x(t) - P^{-1}u^*\| \le \mathrm{e}^{-\sigma t}\cdot \sup_{-\infty<s\le 0}\|x(s) - P^{-1}u^*\|_1, \quad \forall\, t \ge 0,
$$

where σ is the unique positive solution of the equation (12). It should be noticed that $x(t) = P^{-1}u(t)$ is the solution of the system (13) initiated from $x = P^{-1}\phi \in C((-\infty,0],\Omega)$ whenever $u(t)$ is a solution of the system (2) initiated form $\phi \in C((-\infty,0],\Omega)$. Accordingly, we have

$$
\|u(t) - u^*\| \le \mathrm{e}^{-\sigma t}\cdot \sup_{-\infty<s\le 0}\|\phi(s) - u^*\|_1, \quad \forall\, t \ge 0.
$$

We conclude that the system (1) is exponentially stable on Ω and its solutions exponentially decay by (11).

Remark 2 Since the model (1) in [12] is a special case of the model (1), Theorem 2 is an extension of Theorem 3 in [12]. Behavior functions b and time-varying delays $\tau_{ij}(t)$ are required to be differentiable in [9,14-20] and [21,22], respectively. However, the condition on their differentiability is removed in the paper.

Remark 3 Papers [8,18,24,25] are devoted to the stability of neural networks where activation functions are required to be bounded or monotonic. However, the paper makes no any bounded and monotonic assumptions for activation functions.

Remark 4 By stability criterion in [11] we can also derive that the model (1) is exponentially stable under the condition (7). Compared with our method, however, the flaw of the method in [11] is unable to provide the convergent velocity of the solutions.

Example 1 Consider the following neural networks with discrete time-varying and distributed delays

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{dt} = -(2+\mathrm{e}^{u_1(t)})\Big[8u_1(t) - \dfrac{1}{4}f_1\big(u_1(t)\big) - \dfrac{1}{4}g_2\big(u_2(t)-\tau_{12}(t)\big) \\ \qquad -\dfrac{1}{4}\displaystyle\int_{-\infty}^{t} K_{11}(t-s)h_1\big(u_1(s)\big)\mathrm{d}s - \dfrac{1}{5}\displaystyle\int_{-\infty}^{t} K_{12}(t-s)h_2\big(u_2(s)\big)\mathrm{d}s\Big], \\ \dfrac{\mathrm{d}u_2(t)}{dt} = -\big(1+u_2^2(t)\big)\Big[7u_2(t) - \dfrac{1}{8}f_2\Big(u_2(t) - \dfrac{1}{4}g_1\big(u_1(t)-\tau_{21}(t)\big)\Big) \\ \qquad -\dfrac{1}{4}\displaystyle\int_{-\infty}^{t} K_{21}(t-s)h_1\big(u_1(s)\big)\mathrm{d}s - \dfrac{1}{4}\displaystyle\int_{-\infty}^{t} K_{22}(t-s)h_2\big(u_2(s)\big)\mathrm{d}s\Big], \end{cases} \tag{14}$$

where $f_i(r) = g_i(r) = h_i(r) = r - \sin r$, $K_{ij}(r) = \mathrm{e}^{-r}$ and $\tau_{ij}(t) = 2|\sin t|$ for $r \in \mathbb{R}$ and $i,\ j = 1,2$.

The nonnegative function

$$p_{ij}(\beta) = \int_0^{\infty} \mathrm{e}^{\beta s} K_{ij}(s)\mathrm{d}s = \frac{1}{1-\beta}$$

is continuous on $[0,1)$ for $i,\ j = 1,2$. $b = \sup\{\tau_{ij}(t), i,j = 1,2, t \in \mathbb{R}\} = 2$. It is easily verified that model (14) satisfies assumptions (H_1)-(H_4), where $\lambda_1 = \lambda_2 = 4$ and $L(f_i) = 2$ for $i = 1,2$. None of the stability criteria in [8,18,23-25] is applied since activation functions are neither bounded nor monotonic. However, we derive

$$\max_{1\le j\le 2} \frac{1}{\lambda_j}\Big\{L(f_j)\sum_{i=1}^{2}|c_{ij}| + L(g_j)\sum_{i=1}^{2}|d_{ij}| + L(h_j)\sum_{i=1}^{2}|q_{ij}|\Big\} = \max\Big\{\frac{19}{40}, \frac{7}{16}\Big\} = \frac{19}{40} < 1,$$

that is, condition (9) holds for $d_1 = d_2 = 1$. According to Theorem 2, we conclude that the equilibrium point $u^* = (0,0)^T$ of the model (14) is globally exponentially stable and the solution satisfies the following exponential decay estimation

$$|u_1(t)| + |u_2(t)| \le \mathrm{e}^{-\sigma t} \sup_{-\infty < s \le 0} \|\phi(s)\|_1, \quad \forall\, t \ge 0,$$

where $u(t) = (u_1(t), u_2(t))^T$ is any solution of (14) initiated from $\phi \in C((-\infty,0],\mathbb{R}^2)$ and σ is the unique positive solution of the equation

$$\frac{4}{15}\sigma - 1 + \frac{3}{7}\mathrm{e}^{2\sigma} = 0.$$

And a simulation of $u(t)$ is showed in Figure 1.

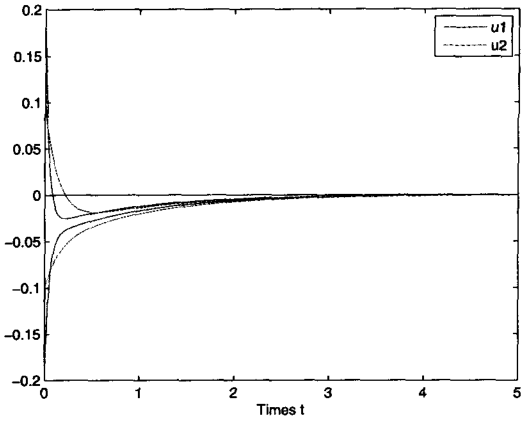

Figure 1: The simulation for the solutions to the neural networks defined in Example 1

5 Conclusions

In this paper, exponential stability of the equilibrium point of a class of neural networks with both discrete time-varying and distributed delays has been investigated. A sufficient condition has been derived for the existence, uniqueness and exponential stability of the equilibrium point. Moreover, it is worth emphasizing that our method can provide the exponentially convergent velocity of the solutions. The proposed criterion is independent of the boundedness, differentiability and monotonicity of activation functions, the symmetry of connected matrices and the differentiability of time-varying transmission delay functions. It means that our criterion is less conservative than some existing ones. Finally, the simple example has demonstrated the effectiveness of our method.

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一类具离散时变时滞和分布时滞神经网络的指数稳定性

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摘 要: 本文讨论一类具离散时变时滞和分布时滞神经网络的指数稳定性。利用非线性测度，本文得到一个与时滞无关的充分条件，它保证了平衡点的存在性、唯一性和指数稳定性。既然新稳定准则不要求激活函数的有界性、单调性及可微性和随时间改变的传递延迟函数的可微性，那么它是某些已有结果的推广。此外，本文的方法的另一个优点是给出了解的指数收敛速度。最后，给出的例子说明我们的方法是有效的。

关键词: 指数稳定性；神经网络；分布时滞；时变时滞；非线性测度